Consider a function $f(x)$ on $[a,b]$. We want to approximate its area under the curve using $n$ subintervals. We are torn between the left or the right endpoint rule. Here is their relationship.

**Regardless of which rule you use,** the length of each subinterval is unchanged:

$$\Delta x = \frac{b-a}{n}$$

**and** the partition

$$P = \{a, a+\Delta x, a+2\Delta x, \dots, a+(n-1)\Delta x, b\}$$

where we notice $b = a + n\Delta x$ (check this!). **Note that every point is expressed in terms of $a$, the starting point, plus some multiples of $\Delta x$.** Thus, we can express every point of $P$ as a sequence

$$c_k = a + k\Delta x, \quad k = 0, 1, 2, \dots n.$$

(check if $c_0 = a$, $c_1 = a + \Delta x$, and so on).

(1) Right endpoint

Now, the right endpoint uses all points except $x = a$. We start with the second term in $P$ and end at $b$. Thus, the area is approximated by

$$\begin{aligned} Area_{\text{right endpoint}}(n) &= \Delta x f(a+\Delta x) + \Delta x f(a+2\Delta x) + \cdots + \Delta x f(a+n\Delta x) \\ &= \Delta x \sum_{k=1}^{n} f(a+k\Delta x) \end{aligned}$$

(you must understand every equality here). Once we are given an interval $[a,b]$ and an $f$, we can express the approximate area under $f$ on $[a,b]$ using the right endpoint rule, in terms of $n$ alone – then taking limits should be straightforward.

(2) Left endpoint

How is this related to left endpoint rule? We have exactly the same setup, except that the points we use now from $P$ is all points except $b = a + n\Delta x$. Therefore, we start with the first term $k = 0$ and add up to the second last term, which is $(a + (n-1)\Delta x)$.

$$\begin{aligned} Area_{\text{left endpoint}}(n) &= \Delta x f(a) + \Delta x f(a+\Delta x) + \cdots + \Delta x f(a+(n-1)\Delta x) \\ &= \Delta x \sum_{k=0}^{n-1} f(a+k\Delta x) \end{aligned}$$

where sanity check checks out (when $k = 0$ we retrieve $f(a)$).

(3) The relationship:

$$Area_{\text{right endpoint}}(n) = \Delta x \sum_{k=1}^{n} f(a+k\Delta x)$$

$$Area_{\text{left endpoint}}(n) = \Delta x \sum_{k=0}^{n-1} f(a+k\Delta x)$$

We note that the right endpoint rule is merely an index shift from the left endpoint rule, with the **same** summand.

**Example 1.** Approximate the area under the graph of the function $f(x) = x^2 - 1$ from $[1,3]$ using $n$ subintervals, with the left and right endpoint rule separately. Show that as $n \to \infty$, both rules yield the same limit.

**Solution.** We identify $a = 1$ and $b = 3$ for the endpoints of our interval. We find the length of the subinterval,

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}.$$

Therefore, our partition of the interval $[1,3]$ is

$$P = \left\{1, 1+\frac{2}{n}, 1+2\cdot\frac{2}{n}, \dots, 1+(n-1)\cdot\frac{2}{n}, 2\right\}$$

and thus each point of the partition is the first point 1 plus some multiple of $\frac{2}{n}$, that is,

$$c_k = 1 + k \cdot \frac{2}{n}, \quad k = 0, 1, 2, 3, \ldots, n.$$

(1) Left endpoint, we use $k = 0, 1, \ldots, n-1$.
(2) Right endpoint, we use $k = 1, \ldots, n$.

Can you complete each part now?